Linear Algebra.

1. R所有由三个实组成的向量 ($[x,y,z]$.)

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x\\y\\0\end{bmatrix}$. ⇒ Matrix A.

3行x3列 3行x1列 = 3行1列.

Nullspace. ⇒ 没有被控制的空间.

$f\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}1\\2\\0\end{pmatrix}$ ⇒ (1,2,3) is mapped to (1,2,0).

↳ NS(A) = {solution to $Ax=0$}

= solution to $f(x,y,z)=0$.

= $\{0,0,z) : z\in R\}$

= z-axis 或 the right space of R^3.

2. Column Space = Image of A : Im(A).

↳ CS(A).

↳ Matrix.

⇒ 一个矩阵的阵能产生的所有输出 (output space).

~~这空间由矩阵的那些列生成.~~

↳ CS(A) = the span of $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix},\begin{pmatrix}3\\6\end{pmatrix}$ ⇒ Basis is $\begin{pmatrix}1\\2\end{pmatrix}$.

given $A=\begin{pmatrix}1&2&3\\2&4&6\end{pmatrix}$. ∴ CS(A) is line $y=2x$ in R^2

~~***~~ 一个Matrix的 column 交换顺序 ⇒ Column space 一样

Summary

3. Basis

↳ 1. Gaussian elimination:

i.e. $\begin{bmatrix}1&2&1\\2&4&3\\1&2&2\end{bmatrix}\begin{matrix}R_1\\R_2\\R_3\end{matrix}$ ⇒ R_1, $R_2' = R_2 - 2R_1$, $R_3' = R_3 - R_1$ ⇒ $\begin{bmatrix}R_1\\R_2'\\R_3'\end{bmatrix}$.

=

3. Basis.

Step 1. Gaussian elimination.

i.e. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix} \Rightarrow \begin{matrix} R_1, \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{matrix}$

$\hookrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3'' = R_3' - R_2'} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Step 2. Find Pivot columns.

$B = \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$ → □ is leading entry (1st)
→ □ is leading entry (3rd)

None.

∴ Pivot columns is column 1 & 3.

Step 3 Back to A.

∵ In B, Pivot columns are 1 & 3

∴ In A, basis are columns 1 & 3

∴ $CS(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}\right\}$

* ① $\dim CS(A) =$ # pivot columns of B.

i.e. In the example above, there are 2 pivot points.

∴ $\dim CS(A) = 2$. ∴ $\dim NS(A) = 3$

∴ column space 有两个独立的方向.

non-pivot columns of B. (5-2=3).

* ② Nullity

$\text{nullity}(A) = \dim NS(A)$

③ Rank.

$\text{rank}(A) = \dim CS(A)$.

一个 m×n 矩阵的 rank

可能为 0, 1 ... min(m, n)

Summary

* NS数量 = Free variable数量.

NS ≠ 0 (A) 且 $Ax = b$ 有解

↓

$Ax = b$ 有无穷解.

✓ 任何span都是 subspace.

1. basis ~~XXXX~~
2. homogeneous system.

reff ✳

i.e. $\begin{bmatrix} 1 & 2 & 4 & 4 \\ 0 & \boxed{-5} & -20 & -11 & -12 \end{bmatrix}$.

↓ 变为1.

$\begin{bmatrix} 1 & 2 & 4 & 4 \\ 0 & 1 & 4 & \frac{11}{5} & \frac{12}{5} \end{bmatrix}$.

↳ $2R_2$

$= 0 \quad 2 \quad 8 \quad \frac{22}{5} \quad \frac{24}{5}$

$\therefore R_1 - 2R_2$

$= 1 \quad 0 \quad -1 \quad -\frac{2}{5} \quad -\frac{4}{5}$

$\therefore \begin{bmatrix} 1 & 0 & -1 & -\frac{2}{5} & -\frac{4}{5} \\ 0 & 1 & 4 & \frac{11}{5} & \frac{12}{5} \end{bmatrix}$. 5.

4. rank-nullity theorm.

⇒ for a $m \times n$ matrix A.

$\text{nullity}(A) + \text{rank}(A) = n$.

Considering num of $\dim NS(A)$ & ~~Rand~~ $\text{rank}(A)$.

~~BOTH~~ Should Fit BOTH: ① $CS(A) \leq 0, 1, \ldots \min(m,n)$

&

② rank-nullity theorm.

Gaussian Elimination Process.

① ~~Find the pivot point in the first row.~~ & 互turn the num below to 0 by + & - the below rows. i.e. $R_2' = R_2 + R_1$. & $R_3' = R_3 - R_1$

② ✳ if ~~no~~ 0 on the most 左上角, then $R_1 \leftrightarrow R_2$ exchange

① ~~Start~~ Start with 左上角 num (if = 0, then exchange 下↔上)

② 把 Pivot 下方全消为0 用. $R_2' = R_2 - \frac{a_n}{a_1} R_1$.

③ 完成第一行一级列后，不要管了，去看 $\begin{matrix} \times & \times & \times & \times \\ \times & \boxed{\times} & \times & \times \\ \times & \times & \times & \times \end{matrix}$.

↳ 0全部在最下面 即 满足 echelon form.

(不行就移/交换行)

Homogeneous & Inhomogeneous ~~function~~ system

① Homo: 右边加完为0. $Ax = 0$.

② Inhomo: Constant solution. / No solution.

↳ ✳ $Ax = b$ have a solution ONLY IF "b is in CS(A)".

✓ ~~Augmented system~~ $\{x \; x \; x \mid 0\}$

为0 ⇒ homo

≠0 ⇒ inhomo.

$\{a \; b \; c \mid d\}$ solution

$a+b+c = d \Rightarrow$ ~~homo~~

$a+b+c \neq d \Rightarrow$ ~~inhomo~~ NO solution

Summary

[In echelon form] to decide.

6. Whether Inhomo is Consistent/Inconsistent?

Step 1 Create augmented matrix.

i.e. $\begin{cases} x_1+x_2+x_3=4 \\ 2x_1+x_2+x_3=5 \\ x_1+3x_2+x_3=6 \end{cases} \Rightarrow \left[\begin{array}{ccc|c} 1&1&1&4 \\ 2&1&1&5 \\ 1&3&1&6 \end{array}\right] = [A|b]$.

* If $m \times n$ matrix A, then augmented is $m \times (n+1)$ (因为多了一列 b).

Step 2. Turn to echelon form. (b列参与)

i.e. $\left[\begin{array}{ccc|c} 1&1&1&4 \\ 2&1&1&5 \\ 1&3&1&6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1&1&1&4 \\ 0&-1&-1&-3 \\ 0&0&-2&-4 \end{array}\right]$.

↳ 无 $(0,0,0|c)$ ※必须为0

∴ 有解.

Step 3 求解.

$-2x_3=-4$. $\quad -x_2-x_3=-3$. $\quad x_1+x_2+x_3=4$.

$x_3=2$. $\quad -x_2-2=-3$. $\quad x_1+3=4$.

$-x_2=-1$ $\quad x_1=1$.

$x_2=1$.

7. Free variables.

i.e. In echelon form

$\left[\begin{array}{ccccc|c} \boxed{1}&0&1&1&&2 \\ 0&\boxed{1}&1&0&&4 \\ 0&0&0&\boxed{1}&0&6 \end{array}\right] \Rightarrow$ Pivot: x_1, x_2, x_4.

∵ Pivot + free = All unknown variables.

∴ free = 5-3=2.

Summary

Transformation.

1. Rotate 90° 逆.

$\cancel{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

↳ $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. ☆

2. Reflection across x-axis.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Multiplication.

1. $(AB)v = A(Bv)$.
2. $AB \Rightarrow$ 先 A (i.e. 旋转) 再 B (反射).
   $BA \Rightarrow$ 先 B 再 A.
3. $(\cancel{A}\, m \times n)(n \times p) = m \times p$ ✓.

Inverse Matrix.

1. A & B are n×n matrices.
   Then $(AB)^{-1} = B^{-1}A^{-1}$
2. $A \cdot A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
3. ~~Inv d~~ ∵ $d = a \cdot c$. ⇒ Where d, a, c are all matrices.
   ∴ inv d = inv c · inv a.

Determinant.

Summary

1. ⇒ $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ (Scalar, no matrix).
2. det of identity matrix = 1.

Nonsingular.

Singular ⇒ No inverse matrix. ∵ $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

i.e. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0$

线性相关 ⇒ Not inversible.

Linear Algebra Unit 1. ①

1. Equations → $Ax=b$ matrices.

i.e. $\begin{cases} 8y-4z=0 \\ x-y-4z=1 \\ -x+5y+2z=0 \end{cases} \Rightarrow \underbrace{\begin{bmatrix} 0 & 8 & -4 \\ 1 & -1 & -4 \\ -1 & 5 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{x\ (\text{vector})} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{b}$.

Functions that can be represented by matrix ↓ linear transformation.

2. $\begin{matrix} & j \\ i & \begin{bmatrix} x & x & x \\ x & x & x \end{bmatrix} \end{matrix}$ m rows by n columns = $m \times n$. ☆

Denote: ith row and jth column by a_{ij}

3. Properties for matrix vector multiplication.

① $A\cdot 0 = 0$.

② $A(cx) = \cancel{c(Ax)}$ $c(Ax)$ (c: scalar)

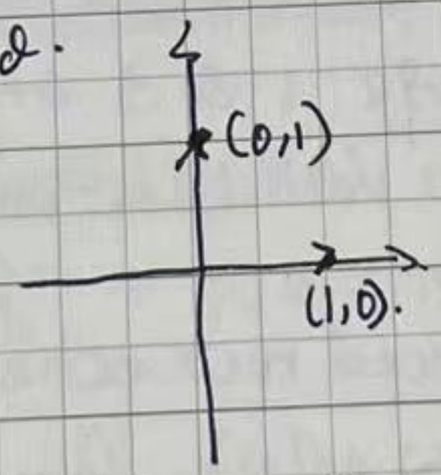
③ $A(x+y) = Ax + Ay$.

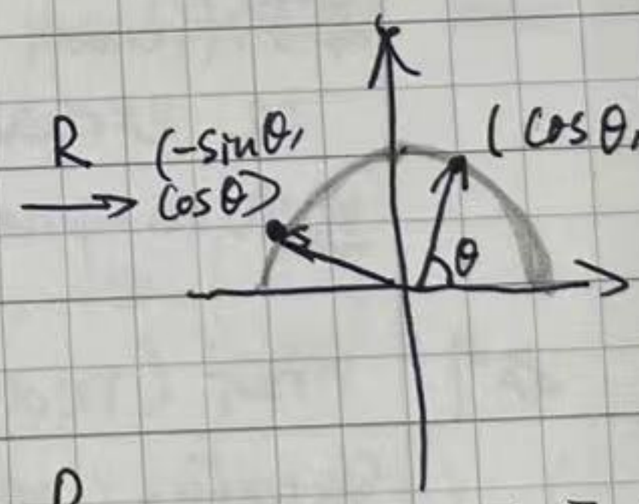
4. Matrices as functions.

"$m \times n$" matrix = a function from R^n to R^m

5. Rotation of matrix.

2d.





$\therefore \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $\xrightarrow{R}$ $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

x y → x y

Summary

$$R\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3d. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} R \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

x y z

MATLAB.

B(2,1).
↓ ↳
Row column.

* Ax=b
↓
MATLAB.
x = A\b.

6. Augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 8 & -4 & 0 \\ 1 & -1 & 4 & 1 \\ -1 & -5 & 2 & 0 \end{array}\right] = [A|b].$$

$$\begin{cases} 8y - 4z = 0. \\ x - y + 4z = 0. \\ -x - 5y + 2z = 0. \end{cases}$$

7. Equation Operations & row operations.
- Multiply equation / row by nonzero ⇒ No change solution.
- Interchange two equation / row ⟹ ↑
- Add, ~~two~~ row to another a multiple of one

8. Gaussian elimination.
Matrix ⇒ Row echelon form.
~~Pivot~~ ~~point~~ 1. Find pivot (左上角的数).
2. If not, 1st & ~~se~~ 2nd row interchange.
3. Make sure zero is at the bottom
* b is in the elimination process.
* $R_2 = \frac{a_{11}}{a_1} R_1$.
Can.
* ~~Different~~ row echelon form for same A.
i.e. exchange 1 & 3 or 1 & 2 row.
* Non-pivot variables use $C_1, C_2 \ldots$ to represent.

#9. rref (reduced row echelon form).
Satisfy: 1. row echelon form.
2. Each pivot is 1.
~~3. Each column, all are 0, except pivot.~~ Each column w/ pivot except one 0, the pivot.

Summary

i.e. $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3. \end{array}$ ✓

↳ ~~$2R_2 - R_1$ (Only change R_1).~~

∴ 找 R_n，然后 $R_1 - R_n'$，R_n 得到为 1 pivot 特点
pivot 为 1

Linear Algebra Unit 1 ②.

1. homogeneous linear system if b is all zero.
↳ solution is all zero.

① Solve homogeneous equations.
- put them into echelon form.
- Let the new echelon form be U, then $Ux=0$ is ~~equiv~~ equivalent to $Ax=0$.

⚠ echelon form w/o pivot's column ⇒ free variable. (let it be other variables i.e. $C_1, C_2 \ldots$)

2. Nullspace.
- Set of all solutions to a homogeneous linear system $Ax=0$

3. Dependent variables
- column w/ pivot.
- Pivot column.

Independent variables -
- Free variables.
- column w/o pivot.

4. m by n matrix ⇒ $Ax=0$ can be at most n dimensional and least ~~use~~ n-m dimensional

5. Vector spaces.
R^n ⇒ N entries ⇒ $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$.

6. $c\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}c\\2c\\3c\end{pmatrix}$. $\begin{pmatrix}1\\2\\3\end{pmatrix} + \begin{pmatrix}-1\\-2\\-3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

Both in the set ⇓ closed under scalar multiplication.

If $Ac\vec{x}_1 = cA\vec{x} = c\vec{0} = \vec{0}$ ⇓ $c\vec{x}_1$ is solution.

Both in the set ⇓ closed under addition.

If $A(\vec{x}_1+\vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{0}+\vec{0}$ ⇓ $\vec{x}_1+\vec{x}_2$ is solution.

Summary

Nullspace of A
⇓
解 $Ax=0$ 方程.

滿足subspace.
① 0 vector
② $u+v$ = ⓦ 滿足
兩 solution.
③ cu = ⓦ 滿足.
any solution
any constant.

Pass through origin.

7. Vector space.
1. ~~vec~~ zero vector
2. ~~cv~~ cv
3. ~~u+v~~ $u+v$ (two solutions) → sum also solution.
} in the ~~vector~~ a set V.

8. Complex vector space C^n is $\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ where z is complex number.

9. Subspaces.
i.e. a vector space inside R^2 = subspace of R^2

Ex. Subspace for R^2.
{ vector 0.
all of R^2.
any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

subspaces for R^3.
{ vector 0.
all of R^3
any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
any plane through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ }.

~~2~~ 2 two entries.
- Dependent: line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- In: The whole R^2.

2 three entries.

1 two entry
- A Line through origin.

3 three entries.
- Dependent: $3v_1+v_2=v_3$ ⇒ 2 ⇒ plane/line. (2 ← independent / 1)
- Independent: The whole R^3

2 three entries.
- Dependent ⇒ 1 ⇒ line.
- In ⇒: plane.

1 three entry.
- line.

Check A independent?
1. A with $\{v_1 \sim v_n\}$
2. nullspace of A.
3. ⇒ only zero in nullspace, then independent.
⇒ non-zero in nullspace, then dependent.

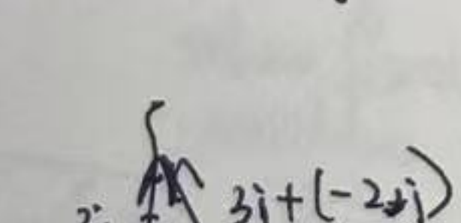
→ echelon form (ref).

Summary

10 span. All linear combination possible.

Span $(v_1 \dots, v_n)$ = {all vectors $c_1v_1 + \dots + c_nv_n$, where $c_1 \sim c_n$ are scalar}

11. i, j.



12 ⚹ Basis: Linearly Independent

MATLAB.

null(A,'r').

rref(A).

13. Every basis for the space has the same number of vectors. → Dimension.

14. R^n dimension n.

15. R^n dimension n ⇒ Constraint: i.e. $x=-3y$ (line).
↳ ∴ dimension: 1
- Basis 有幾多個 vector ⇒ 咁 dimension

★ i.e. $x+y+z=0$ in R^3 have how many dimensions?

① $x=-z-y$ ⇒ two variables.

② ∴ $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-z-y\\y\\z\end{bmatrix}$ ③⇒ $\begin{bmatrix}-z\\0\\z\end{bmatrix}+\begin{bmatrix}-y\\y\\0\end{bmatrix}$ ④⇒ $z\begin{bmatrix}-1\\0\\1\end{bmatrix}+y\begin{bmatrix}-1\\1\\0\end{bmatrix}$

⇓ two dimensions.

★★ Theorm: n dimension, if gave more than n vectors, then their relation must be linear.

★★ Rank-Nullity ∴ $R^{[n]}$ − independent 嘅 vectors = ~~free variables~~ Degree of freedom.

↳ i.e. Dimension of R^4 span$\left[\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix},\begin{pmatrix}-2\\2\\-2\\2\end{pmatrix}\right]$

↳ $-2\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}=\begin{bmatrix}-2\\2\\-2\\2\end{bmatrix}$ ⇒ Linear ⇒ 1.

16. ★ dim NS(A) = # non-pivot columns of B (B 係 A 嘅 echelon form)
& NS(A) = NS(B)
↓ free variable.

★ Find echelon form before everything.

Summary

Basis: ① echelon.
② non-pivot = dim NS(A).
③ Back sub to find x in Bx=0. (B 係 echelon.)
④ list is basis. (the)

★ Ax=0 不零嘅例子 − 梗 echelon.
Ax=b 零.

2026 · 8 · 4

Identity matrix.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ⇒ Diagonal 1, others 0

"I" be Identity matrix., then $AI = IA = A$.

Rank: 能提供多少个独立条件.

- Rank-Nullity Theorem.

Basis: 1. $\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$ → echelon form → pivots. → in columns 1 & 3. → .

NS 2. dim NS(A) = non-pivots. (Nullity)

CS. ~~AIS~~ ~~3~~. 3. dim CS(A) = pivot. (Rank)

~~3. dim CS(A) =~~

~~E~~S.* 4. Column = dim NS(A) + dim CS(A).

= ~~Inhomo have solution~~:

Inhomo equation.

~~∴ rank = null~~

① 当 rank = Column num

∴ ~~rank~~ 每一列都有 pivot, 没 free var

∴ 1. b in column space

几个解: b 是否满足 A 的线性解.

- 是 ⇒ Column num - rank ≠ 0 ⇒ infinite.
  Column num = rank ⇒ 1 (只有在 column space 中才可以)
  column num = rank = 行数 ⇒ 1 (所有 b)
- 否 ⇒ 0 个解 ~~无解~~

(其余无解)

* rank = column num 但 b 不知道满足不满足 A 的线性解 ⇒ 0/1.

Summary

Identity Matrix特点.

① $Ix=b$.

即是一般的 $x=b$.

② $\cdot IA=A=AI$.

③ 每行每列都有 pivot.

④ $I^{-1}=I$.

Inverse matrices.

1. Square. (Invertible 情况下).

$A^{-1}\cdot A=I.=A\cdot A^{-1}$

↳ Invertible ⇒ nonsingular

E.g. $R=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ ⇒ inverse. $R^{-1}=\begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix}$.

Summary

2026 · 8 · 2.

* Free variable = no pivot.

Column space.

= All linear combinations of a matrix.

Can solve $Ax=b$ exactly when b is in C(A).

i.e. [Augmented] b_1, b_2-b_1, $b_3-b_2-b_1$ — echelon → $\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore b_3-b_2-b_1=0$.

$\therefore$ Any $f(n) \Rightarrow$ substitute.

① If a comb of rows of A gives a zero row, then same comb of entries of b must be 0.

i.e. $A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$, $x\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

↳ ~~solvable~~ : $R_1+R_2-R_3=0$.

$\therefore b_1+b_2-b_3=0$.

$b_1+b_2=b_3$.

$\therefore$ Solvable: $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$ b_1, b_2, b_3 $\quad b_1+b_2=b_3$. $\quad 2+3=5$.

Unsolvable: $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ b_1, b_2, b_3 $\quad b_1+b_2=b_3$. $\quad 2+3\neq 6$.

[No need echelon].

② Echelon $\Rightarrow \left[\begin{array}{ccc|c} x & x & x & x \\ 0 & 0 & 0 & \text{nonzero} \end{array}\right]$

$\Downarrow$

Inconsistent.

$\Downarrow$

No solution.

Summary

5 . . E.g. $x-2y-2z=b_1$ $2x-5y-4z=b_2$ $4x-9y-8z=b_3$

$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 2 & -5 & -4 & b_2 \\ 4 & -9 & -8 & b_3 \end{array}\right] \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

① R_2-2R_1 :
$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & b_2-2b_1 \\ 4 & -9 & -8 & b_3 \end{array}\right]$$

② R_3-4R_1 :
$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & b_2-2b_1 \\ 0 & -1 & 0 & b_3-4b_1 \end{array}\right] \begin{matrix} R_1' \\ R_2' \\ R_3' \end{matrix}$$

③ $R_3'-R_2'$:
$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_3-b_2-4b_1-b_2+2b_1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & -1 & 0 & b_2-2b_1 \\ 0 & 0 & 0 & b_3-b_2-2b_1 \end{array}\right]$$

1. $b_3-b_2-2b_1=$ nonzero → no solution.
2. $b_3-b_2-2b_1=0$ → infinite.

④
$$\left[\begin{array}{ccc|c} 1 & -2 & -2 & b_1 \\ 0 & 1 & 0 & 2b_1-b_2 \\ 0 & 0 & 0 & b_3-b_2-2b_1 \end{array}\right]$$

⑤
$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & -2 & b_1-2(2b_1-b_2) \\ 0 & \textcircled{1} & 0 & 2b_1-b_2 \\ 0 & 0 & 0 & b_3-b_2-2b_1 \end{array}\right]$$
$\underline{x}$ $\underline{y}$ z

∴ z is free variable ; $z=c_1$.

[Set one free 1 and others 0 ; each variable/ each equation]

Summary

① $Ax=0$ ⟨set all free $z=1$.

$x-2=0$.

$y=0$.

$x_1=\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

② All solutions :

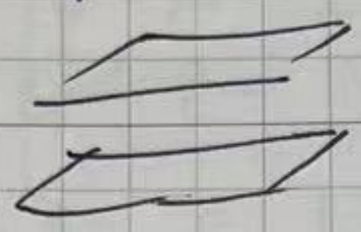
$\vec{x}=\vec{x}_{particular}+C\cdot\vec{x}_{special}$

↓ constant.

Eg. ① $x-5y+2z=9$ is a plane in $\mathbb{R}^3$

② $x-5y+2z=0$ is a plane in $\mathbb{R}^3$

1. Parallel.



2. Intersect.



$\because q \neq 0$.

$\therefore$ Parallel.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} q \\ 0 \\ 0 \end{bmatrix} + C_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

P_0

① $\because$ All Point of S have. this form.

$\therefore P_0$ is in S. $\therefore x_1 = q$.

& S: $x-5y+2z=q$.

$$\therefore \begin{cases} x-5y+2z=q. \\ y=0 \\ z=0. \end{cases}$$

② $y=1, z=0$. $\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} q \\ 0 \\ 0 \end{bmatrix} + C_1\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

$\therefore x_2 = 5$.

③ $y=0, z=1$ $+ C_2\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

$\therefore x_3 = -2$.

Summary

Inhomogeneous linear System.

$Ax=b$. ⇒ ① No Solution.

② . Solution ⇒ ① Nullspace ($Ax=0$).

② Any one solution.

* x_p solution to $Ax=b$. & x_n is solution to $Ax=0$.

THEN: $x = x_p + x_n$ is a solution to $Ax=b$.

→ Proof: ∵ $Ax_p=b$ & $Ax_n=0$.(add).

∴ $A(x_p+x_n)=b$ ←

E.g.

$$x = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix} + C_1 \begin{bmatrix} -8 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + C_2 \begin{bmatrix} -4 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

particular solution.

general homogeneous solution.

Composition & Matrix ≠ multiplication.

90° counterclockwise. × $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Reflect across x-axis × $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

* $(AB)v = A(Bv)$.

Summary

~~Eig~~ Eigenvalue & vector.

本來假設只滿足于 square matrix. $(n \times n)$.

* Eigenvalue of A is a scaler λ that $Av = \lambda v$
* Eigenvector associated with ~~an~~ λ is a nonzero vector v such that $Av = \lambda v$

* zero vector is never an eigvector.
* Everyone allows $\lambda = 0$.

Eg. ① $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 2 & 1 & 0 \end{bmatrix}$ & $V = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$.

$\therefore A \cdot V = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix}$.

$\because Av = \lambda v$

$\therefore \lambda = 3 \left(\because 3 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix} \right)$.

② If $\underbrace{w}_{\text{vector}} = 2v$, eigenvalue still same?

$Aw = A \cdot (2v) = 2Av$

$\because Av = \lambda v = 3v$

$\therefore Aw = 2Av = 6v$

$\because w = 2v$.

$\therefore Aw = 3w$

$\therefore$ eigenvalue is still the same.

Summary

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

= ① $\begin{bmatrix} \textcircled{a} & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ — ② $\begin{bmatrix} a & \textcircled{b} & c \\ d & e & f \\ g & h & i \end{bmatrix}$ + ③ $\begin{bmatrix} a & b & \textcircled{c} \\ d & e & f \\ g & h & i \end{bmatrix}$.

i.e. $a(ei-hf)$

$\begin{bmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & i-\lambda \end{bmatrix}$.

Determinant.

$\det(A)$ 指一个matrix对应的一个数.

E.g. $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(A) = ad-bc$.

A: ✱ $\det(A) \neq 0$, 可逆, 满秩(Full-rank).

$\det(A) = 0$, 不可逆, Colums 线性相关

~~Det~~ Characteristic Poly.

✱ Eigenvalues 是 characteristic poly 的根

E.g. $P(\lambda) = \det(A-\lambda I)$.

i.e. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

then $A-\lambda I = \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$.

$P(\lambda) = (2-\lambda)^2 - 1$.

let $P(\lambda) = 0$, then 求根 will be eigvalue

Result: 1 & 3.

Eigenspace.

E.g. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - 3 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

For 3: A $\lambda \cdot I$

$x = y$

Eigenspace = $\text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$.

Trace 为对角线之和.

Summary

$A = PDP^{-1}$ diagonalization.

ODE. Ordinary Differential Equation.

1. $Ve^{\lambda t}$ is a solution, if & only if v is an eigenvector of A with for $\dot{x}=Ax$.
eigenvalue λ.

Proof: $x=ve^{\lambda t}$, $\dot{x}=\lambda Ve^{\lambda t}$

$\because \dot{x}=Ax$

$\therefore \lambda Ve^{\lambda t}=AVe^{\lambda t}$

$\lambda V=AV$

2. $\dot{x}=Ax$

1. $\det(\lambda I-A)=0$. Find eigenvalues.
2. $(\lambda I-A)v=0$ Find eigenvectors (normal mode.)
3. ~~每一组 $(\lambda I-A)$~~ (λ, v) 产生一个解. $x(t)=ve^{\lambda t}$
4. $n\times n$ 找到 n 个 independent eigenvectors

则为 $x(t)=\underbrace{\sum_{i=1}^{n} C_i v_i e^{\lambda_i t}}$

basis solution.

* 5 如果求其用(word problem), 则求出解.

每一个环节都不能少.

Eg. $A=\begin{bmatrix}-1&1&0\\1&-2&1\\0&1&-1\end{bmatrix}$ → Fundamental of Matrix.

~~$x(t)=X(t)C$, $C=\begin{bmatrix}C_1\\C_2\end{bmatrix}$~~.

$X(t)=\begin{bmatrix}x_1^1(t) & x_2^1(t)\\ x_1^1 & x_2^1\end{bmatrix}$.

1. $\det(\lambda I-A)=0$.

$\lambda I=\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$

$\therefore \begin{bmatrix}\lambda+1&-1&0\\-1&\lambda+2&-1\\0&-1&\lambda+1\end{bmatrix}$.

Summary

小求 det 得 $\lambda_1=-1, \lambda_2=0$,

$\lambda_3=-3$.

2. ① $\lambda=0$ ② $\lambda=-3$ ③ $\lambda=-1$

代入求 $V=\begin{bmatrix}x\\y\\z\end{bmatrix}$. 得 $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ $\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

得 $V=t\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}$

3. $\because \begin{bmatrix}1\\1\\1\end{bmatrix}e^{0t}=\begin{bmatrix}1\\1\\1\end{bmatrix}$. ✓

且 $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ & $\begin{bmatrix}1\\0\\1\end{bmatrix}\Rightarrow 0$.

$\begin{bmatrix}1\\-2\\1\end{bmatrix}e^{-3t}$ $\begin{bmatrix}1\\0\\1\end{bmatrix}e^{-t}$

Companion System.

將高階ODE ⇒ 多个一階的ODE ⇒ matrix

E.g. $\dddot{x}+5\ddot{x}+6\dot{x}=0$ to a first-order system of ODEs.

let $y=\dot{x}$, then ~~$\dot{y}+5y+$~~ $\ddot{x}=-5\dot{x}-6x$.

$\therefore \dot{y}=-5y-6x$.

$\because \dot{x}=Ax$, then $A=\begin{bmatrix}0&1\\-6&-5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$ $\hookrightarrow \begin{cases}\dot{x}=y\\ \dot{y}=-5y-6x.\end{cases}$

E.g. Companion matrix.

i.e. $A=\begin{bmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&0&\ddots&0\\-a_0&-a_1&-a_2&&-a_{n-1}\end{bmatrix}$.

$z=\begin{bmatrix}x_0\\x_1\\\vdots\\x_{n-1}\end{bmatrix}=\begin{bmatrix}x\\\dot{x}\\\ddot{x}\\\vdots\\x^{n-1}\end{bmatrix}$.

① $\dddot{x}+2\ddot{x}+3\dot{x}+4x=0$.

let $x_0=x$, $x_1=\dot{x}$, $x_2\doteq\ddot{x}$, ~~$x_3=\dddot{x}$~~

$\therefore \dot{x}_0=\dot{x}=x_1$., & $\dot{x}_1=\ddot{x}=x_2$

$\because \dddot{x}=-2\ddot{x}-3\dot{x}-4x$ $=-4x_0-3x_1-2x_2$.

~~$=-2x_2$~~ $=-2x_2-3x_1-4x_0$

$\therefore \begin{bmatrix}\dot{x}_0\\\dot{x}_1\\\dot{x}_2\end{bmatrix}=$ ~~$\begin{bmatrix}-2&0&0\end{bmatrix}$~~ $\begin{bmatrix}0&1&0\\0&0&1\\-4&-3&-2\end{bmatrix}$.

Summary

* $F_{damper} = b_1 \dot{x}_1$
  - b_1: damper coefficient
  - $\dot{x}_1$: speed.

振动系统 $\lambda = i\alpha$.

↳ $e^{\lambda t} = e^{i\alpha t}$

$e^{i\alpha t} = \cos(\alpha t) + i\sin(\alpha t)$.

- Companion System of the coupled oscillator.

let $\frac{k}{m} = w^2$, $\ddot{x} = Bx$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $B = w^2 \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

↳ Convert to first order.

let ~~$y_1^* =$~~ $y_1 = \dot{x}_1$, $y_2 = \dot{x}_2$

$\therefore y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \dot{x}$

$\because \ddot{x} = Bx$

$\therefore \dot{y} = Bx$.

$\because B = w^2 \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$, 且 $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$\therefore \dot{y} = w^2 \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$\begin{cases} \dot{x} = y \\ \dot{y} = Bx \end{cases}$ (前两排)

$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2w^2 x_1 & w^2 x_2 & 0 & 0 \\ w^2 x_1 & -2w^2 x_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix}$.

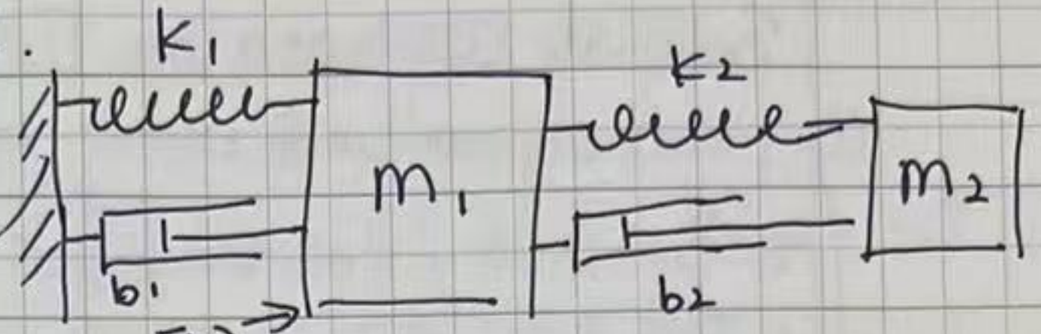
Summary

$\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$ Block Matrix.

1.

$F_{damper} = b_1 \cdot \dot{x}_1$

Eg.



① Equation of motion.

Mass 1: [k_1, b_1, k_2, b_2, F].

$m_1\ddot{x}_1 = -k_1x_1 - b_1\dot{x}_1 + k_2(x_2 - x_1) + b_2(\dot{x}_2 - \dot{x}_1) + F$

Mass 2: [k_2, b_2].

$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1)$.

F=ma.

② Turn to Matrix (high ODE).

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{matrix} m_1: -k_1x_1 - b_1\dot{x}_1 + k_2x_2 - k_2x_1 + b_2\dot{x}_2 - b_2\dot{x}_1 + F \\ m_2: -k_2x_2 + k_2x_1 - b_2\dot{x}_2 + b_2\dot{x}_1 \end{matrix}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -(b_1+b_2) & b_2 \\ b_2 & -b_2 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Let $y_1 = x_1$, $y_2 = \dot{x}_2$, $\dot{x}_1 = y_1$, $\dot{x}_2 = y_2$

③ Turn to Matrix (Low order ODE).

(前两排).

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} & \frac{-(b_1+b_2)}{m_1} & \frac{b_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} & \frac{b_2}{m_2} & \frac{-b_2}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{F}{m_1} \\ 0 \end{bmatrix}$$

Summary

(除 m_1, 除 m_2).

$$\Rightarrow \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{-(b_1+b_2)}{m_1} & \frac{b_2}{m_1} \\ \frac{b_2}{m_2} & -\frac{b_2}{m_2} \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{-(k_1+k_2)}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{F}{m_1} \\ 0 \end{bmatrix}.$$

(后两排).

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{F}{m_1} \\ 0 \end{bmatrix}.$$

Solve the equation on [1].

① Find eigenvalue & vector of $\begin{pmatrix} 0 & I \\ B & 0 \end{pmatrix}$.

$\because Bv = \lambda v$, let $v = \begin{bmatrix} x \\ y \end{bmatrix}$.

$\therefore \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$.

$$\begin{bmatrix} y \\ Bx \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$\begin{cases} y = \lambda x \\ Bx = \lambda y \end{cases} \Rightarrow Bx = \lambda \cdot \lambda x = \lambda^2 x$$

$\therefore x$ is eigenvector of B,
λ^2 is eigenvalue of B.

$\therefore \begin{bmatrix} x \\ \lambda x \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}$.

If B has two eigenvalues and vectors, then let it be λ_1^2 and λ_2^2

$\therefore$ eigenvalues are: $-\lambda_1, -\lambda_2, \lambda_1, \lambda_2$.

$\therefore$ eigenvectors are: $\begin{bmatrix} -\lambda_1 \\ -\lambda\lambda_1 \end{bmatrix}, \begin{bmatrix} -\lambda_2 \\ -\lambda\lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda_1 \\ \lambda\lambda_1 \end{bmatrix}, \begin{bmatrix} \lambda_2 \\ \lambda\lambda_2 \end{bmatrix}$

$\therefore \begin{bmatrix} v_1 \\ -\lambda v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ -\lambda v_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ \lambda v_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ \lambda v_2 \end{bmatrix}$.

* 記住.

② let $A = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} w^2$.

Eigenvalue: $\det(\lambda I - A)$.

$\therefore \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} w^2 = \begin{bmatrix} \lambda+2 & -1 \\ -1 & \lambda+2 \end{bmatrix} \cdot = \begin{bmatrix} \lambda+2w^2 & -w^2 \\ -w^2 & \lambda+2w^2 \end{bmatrix}$.

~~$\det \begin{bmatrix} \lambda+2 & -1 \\ -1 & \lambda+2 \end{bmatrix} w^2 \cdot = (\lambda+2)^2 - 1 = \lambda^2+4+4\lambda-1 = \lambda^2+4\lambda+3$~~

$= \det \begin{bmatrix} (\lambda+2)w^2 & -w^2 \\ -w^2 & (\lambda+2)w^2 \end{bmatrix} = (\lambda+2)^2 w^4 - w^4 = (\lambda^2+4+4\lambda)w^4 - w^4$

$= \det \begin{bmatrix} \lambda+2w^2 & -w^2 \\ -w^2 & \lambda+2w^2 \end{bmatrix} = (\lambda+2w^2)^2 - w^4$

$= \lambda^2 + 4w^4 + 4\lambda w^2 - w^4$.

$= \lambda^2 + 3w^4 + 4\lambda w^2$.

$= (\lambda + w^2)(\lambda + 3w^2)$

$\lambda^2 = \mu$

Summary

$\lambda_j = -w_j^2$

$w_j = \sqrt{-\lambda_j}$

↑

解特殊矩阵 A eigenvalue.

③ let $(\lambda+w^2)(\lambda+3w^2)=0$.

$\therefore \lambda_1 = -w^2$, $\lambda_2 = -3w^2$.

∴ Eigenvalue of B is $-w^2$ & $-3w^2$.

∠ Eigenvalue of A is $\sqrt{B}$

∴ A is $\pm iw$ $\pm\sqrt{3}iw$.

④ Eigenvectors. for B.

~~for $-w^2$~~

∵ $(\lambda I - B)v = 0$.

$\therefore \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}-2w^2 & w^2\\ w^2 & -2w^2\end{bmatrix} = \begin{bmatrix}\lambda+2w^2 & -w^2\\ -w^2 & \lambda+2w^2\end{bmatrix}$.

1. For $-w^2=\lambda$

$\therefore = \begin{bmatrix}-w^2+2 & -1\\ -1 & -w^2+2\end{bmatrix} = \begin{bmatrix}-w^2+2w^2 & -w^2\\ -w^2 & -w^2+2w^2\end{bmatrix} = \begin{bmatrix}w^2 & -w^2\\ -w^2 & w^2\end{bmatrix}$

$\therefore = \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$. $\therefore \begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ (Find nullspace)

$\therefore v_1 - v_2 = 0$, $-v_1 + v_2 = 0$. $\therefore v_1 = v_2$ $\therefore v = \begin{bmatrix}1\\ 1\end{bmatrix}$.

2. For $-3w^2=\lambda$.

$= \begin{bmatrix}-3w^2+2w^2 & -w^2\\ -w^2 & -3w^2+2w^2\end{bmatrix} = \begin{bmatrix}-w^2 & -w^2\\ -w^2 & -w^2\end{bmatrix} = \begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}$.

∴ Nullspace : $\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$.

$-v_1 - v_2 = 0$.

$-v_1 = v_2$

$\therefore v = \begin{bmatrix}1\\ -1\end{bmatrix}$. → Nullspace.

Summary

For A matrix:

∴ for $-w^2$ eigenvector.

$\begin{bmatrix}x_1\\ x_2\\ y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1\\ 1\\ \pm iw\\ \pm iw\end{bmatrix}$

∴ for $-3w^2$

$\begin{bmatrix}x_1\\ x_2\\ y_1\\ y_2\end{bmatrix} = \begin{bmatrix}1\\ -1\\ \pm\sqrt{3}iw\\ \mp\sqrt{3}iw\end{bmatrix}$.

displacement
displacement.
speed
speed

⑤ Complex solution · (General).

$= e^{i\omega t}\begin{pmatrix}1\\1\\i\omega\\i\omega\end{pmatrix}, e^{i\sqrt{3}\omega t}\begin{pmatrix}1\\-1\\i\sqrt{3}\omega\\-i\sqrt{3}\omega\end{pmatrix}$, and complex conjugates.

∴ General solution = linear combination of theses.

$e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ ✓.

⑥ In physics world.

Only take Real part: $e^{i\omega t}\begin{pmatrix}1\\1\end{pmatrix} = \begin{bmatrix}\cos(\omega t)+i\sin(\omega t)\\ \cos(\omega t)+i\sin(\omega t)\end{bmatrix}$.

1. $= \begin{bmatrix}\cos(\omega t)\\ \cos(\omega t)\end{bmatrix}$. $= \begin{bmatrix}\sin(\omega t)\\ \sin(\omega t)\end{bmatrix}$.

↑ Real ↑ ~~Imag~~ Imag.

∴ $\begin{bmatrix}\cos(\omega t)\\ \cos(\omega t)\end{bmatrix}, \begin{bmatrix}\sin(\omega t)\\ \sin(\omega t)\end{bmatrix}$

2. $e^{i\sqrt{3}\omega t}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}\cos(\sqrt{3}\omega)+i\sin(\sqrt{3}\omega)\\ -\cos(\sqrt{3}\omega)-i\sin(\sqrt{3}\omega)\end{bmatrix}$.

$= \begin{bmatrix}\cos(\sqrt{3}\omega)\\ -\cos(\sqrt{3}\omega)\end{bmatrix}$. $= \begin{bmatrix}\sin(\sqrt{3}\omega)\\ -\sin(\sqrt{3}\omega)\end{bmatrix}$

↑ Real ↑ Imag.

*Imag 在这里可以代表初始速度.

⑦ General solution.

$\chi_1 = A_1\cos(\omega t+\phi_1) + A_2\cos(\sqrt{3}\omega t+\phi_2)$

$\chi_2 = A_1\cos(\omega t+\phi_1) \pm A_2\cos(\sqrt{3}\omega t+\phi_2)$

Summary

Fundamental Matrix.

① $|\vec{X}|$ (Det) $\neq 0$.

② $\dot{X} = AX$ (columns solve the system.).

③ Can be unfixing way

④ $[x_1 \; x_2]$ ~~two~~ ~~of~~ every column - independent.

⑤ $n \times n$.

- General ~~system~~ → solution → solution.

① $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = [\vec{x}_1 \; \vec{x}_2]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

② $\vec{x} = X\vec{c}$ (general solution.

- Look like?

i.e. $[X\vec{c}_1 \; X\vec{c}_2] = X[\vec{c}_1 \; \vec{c}_2]$.

most general $= XC$, $|C| \neq 0$.

Summary

Project. 1.

✱ ~~Bla~~ Balance in ecosystem (exsist ? stable ?).

① logistic's simply model.

$\dot{x} = k(x) \cdot x \Rightarrow$ $x=0$ → 完全灭绝. / $x=35$ → Resistant. → 出生/死亡平衡.

k(x): increasing. ~~Speed~~.

② Lotka-Volterra ~~relevance~~ model

↳ 列出非线性微分方程组.

i.e. $\begin{cases} \dot{H} = H(a-L) \\ \dot{L} = L(H-1) \end{cases}$ (a: increasing population of 1.)

L: Population 2. H: Population 1.

↳ Solve: 得出互相影响后的平衡点.

③ Jacobian & eigenvalue.

- x^* near balance point.
- u is the least 扰动.

$\dot{u} \approx J(x^*)u$.

$\therefore \dot{u} = Au$.

$\Rightarrow$ $Re(\lambda)<0$, 扰动逐消失.
$Re(\lambda)>0$, 增长.
$Re(\lambda)=0$ 无法判断.
既有正负: 不同方向 时间行为不同.

Summary

① . . .

* $\dot{x} = (a - bx) \cdot x$.

[$\dot{x} > 0 \Rightarrow$ increase growth rate.

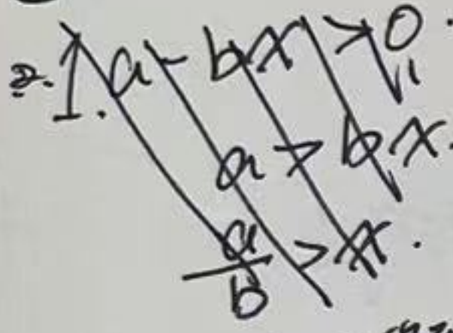
$\dot{x} < 0 \Rightarrow$ decrease growth rate]

∵ $x > 0$ ∴ $(a - bx)$ control sign.

~~or~~

~~both~~

[已知 a, b 都为 +].



∵ $\dot{x} = 0$ 为平衡.

∴ $a - bx = 0$.

$a = bx$

$\frac{a}{b} = x$.

* $\frac{\partial f}{\partial x}$ = x改变对 f 增长率影响.

$\frac{dx}{dt}$ = x 随时间的变化速度

∫

② Add another y specie.

↳ also $\dot{y} = (d - ey) \cdot y$

∵ y appear ⇒ affect x.

∴ $\dot{x} = (a - bx - cy) \cdot x$

∵ x affect y

∴ $\dot{y} = (d - ey - fx) \cdot y$.

$= (a - bx)x - cxy$.

$= \underbrace{(d - ey)}_{\text{logistic}} y - \underbrace{fxy}_{\text{interaction}}$.

Logistic: $k(x) = a - bx$.

∵ $\dot{x} = \underbrace{k(x)}_{\text{growth rate per species}} \cdot \underbrace{x}_{\text{specie number}}$

∴ $\dot{x} = (a - bx) \cdot x$ → only 1 specie.

$= ax - bx^2$

Lotka-Volterra.

⇓

i.e. for interaction: ① c & $f > 0 \Rightarrow$ negative inter.

∴ competition.

② $c > 0$ & $f < 0 \Rightarrow$ cause x to decrease & benefit y

∴ x is prey

y is predator.

② . $c < 0$ & $f < 0 \Rightarrow$ mutualism.

Summary

Linearization 使用 Jacobian matrix.

∵ $\begin{cases} \dot{x} = ax - bx^2 - cxy \\ \dot{y} = dy - ey^2 - fxy. \end{cases}$ 非线性.

为了变成 linear.

∴ Jacobian matrix.

[tangent-line approximation]. 共生.

∵ $f(x) \approx f(a) + f'(a)(x - a)$.

但 function 和 x, y 都有关系.

∴ $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.

∵ $f_x = \frac{\partial f}{\partial x}$ & $f_y = \frac{\partial}{\partial y}$

只改 x 只改 y

let $f_x = \frac{\partial f}{\partial x}$

$f_y = \frac{\partial f}{\partial y}$

$g_x = \frac{\partial g}{\partial x}$

$g_y = \frac{\partial g}{\partial y}$.

Jacobian matrix.

$J(x,y) = \begin{bmatrix} & \\ & \end{bmatrix}$

∵ (泰勒) $f(x) \approx f(a) + f'(a)(x-a)$.

∴ $f(x,y) \approx f(a,b) + f_x(x-a) + f_y(y-b)$.

$g(x,y) \approx g(a,b) + g_x(x-a) + g_y(y-b)$.

∵ $f(x,y) = \dot{x}$ $g(x,y) = \dot{y}$

matrix: $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \approx \begin{bmatrix} f(a,b) \\ g(a,b) \end{bmatrix} + \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} \vec{a} \begin{bmatrix} \vec{x} - \vec{a} \end{bmatrix}$.

plug (a,b) in (x,y) once partial derivative first look.

let $\vec{a} = \begin{bmatrix} a \\ b \end{bmatrix}$. $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

↓.

let: $\vec{f(x)} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ $\vec{J} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$.

$\vec{f(a)} = \begin{bmatrix} f(a,b) \\ g(a,b) \end{bmatrix}$.

∴ $J(a)$ is a matrix of partial derivative.

↳ Jacobian matrix.

[stability: evaluate matrix at critical point].

→ eigenvalue at point a has neg real ⇒ critical stable

→ positive real ⇒ unstable

→ highest = 0 → × determined.

Summary

1.

Physical meaning:

i.e. Hare + Lynx equations

Hare / Lynx. $\begin{cases} \dot{x} = (a-bx)x - cxy \\ \dot{y} = (d-hy)y - fxy \end{cases}$

① $y=0 \Rightarrow x$ increase exponentially then. if $a>0$.

[$b=0$, because bx is other limiting factors].

② $x=0 \Rightarrow y$ decrease exp then if $d<0$.

[$h=0$, ...].

③ ∵ Lynx feat Hare.

∴ $c>0$, $f<0$.

for in $\dot{x}$, $a>0$ $b=0$
in $\dot{y}$, $d<0$ $h=0$.
$c>0$
$c<0$

↳ ∴ ⇒ $\begin{cases} \dot{H} = aH - HL. \\ \dot{L} = -L + HL \end{cases}$

[Qu] 2. Find: Critical point: (0,0) & (1,a).

what happens?

3. ① linearize: Near point: $H = 1+h$.

[New point: (h,l)] $L = a+l$.

$\therefore \dot{H} = \dot{h}$

$\dot{L} = \dot{l}$

∵ $\begin{pmatrix} h' \\ l' \end{pmatrix} = A \begin{pmatrix} h \\ l \end{pmatrix}$ → $\therefore A = \begin{bmatrix} 0 & -1 \\ a & 0 \end{bmatrix}$.

↓ New Jacobian matrix.

∵ $J = \begin{bmatrix} \frac{\partial \dot{H}}{\partial H} & \frac{\partial \dot{H}}{\partial L} \\ \frac{\partial \dot{L}}{\partial H} & \frac{\partial \dot{L}}{\partial H} \end{bmatrix}$. ∴ $J_{(H)} = \begin{bmatrix} a-L & -H \\ L & -1+H \end{bmatrix}$

∵ $\lambda^2 + a = 0$.

$\lambda = \pm i\sqrt{a}$. ∴ $J(1,a) = \begin{bmatrix} a-a & -1 \\ a & -1+1 \end{bmatrix}$

(h,l)

$= \begin{bmatrix} 0 & -1 \\ a & 0 \end{bmatrix}$.

∵ $H = h$, $L = l$;

$\begin{cases} \dot{H} = aH - HL. \\ \dot{L} = -L + HL. \end{cases}$

∴ $\dot{h} = a(1+h) - (H+h)(a+l)$.

$\dot{h} = a + ah - (a + l + ah + hl)$.

$\dot{h} = a + ah - a - l - ah - hl$.

$= -l - hl$.

∴ 以下 ÷ 忽略.

Summary

∴ $\dot{h} \approx -l$.

* λ 的实：振幅.

虚：frequency.

↙ Matrix A.

Deriving nonlinear DE center.

e.g. $\dot{x} = -x + xy$ $\Rightarrow$ eliminate t.

$\dot{y} = y - xy$

$\therefore \frac{dy}{dx} = \frac{y(1-x)}{x(y-1)}$.

$dy \cdot x(y-1) = dx \cdot y(1-x)$.

$\frac{dy \cdot (y-1)}{y} = \frac{dx(1-x)}{x}$.

$\frac{y-1}{y} dy = \frac{1-x}{x} dx$.

$(1 - \frac{1}{y}) dy = (\frac{1}{x} - 1) dx$. ↓Integral.

$y - \cancel{\log y}\, \ln y = \cancel{\log}\, \ln x - x + C_1$

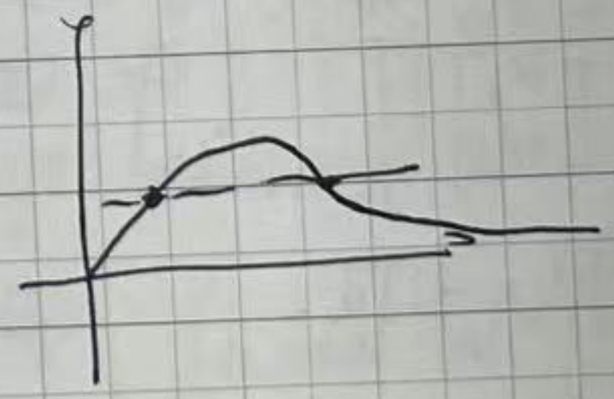
两边取对数. $e^{y - \ln y} = e^{\ln x - x + C_1}$.

$\frac{e^y}{y} = \frac{x}{e^x} \cdot C_2$.

$\cancel{e^x} \cdot \frac{x}{e^x} \cdot \frac{y}{e^y} = C$.

↳





Summary

$\det(A-\lambda I)=0$ ①

② $\text{Det}\,A=\begin{bmatrix}-\lambda & -1\\ a & -\lambda\end{bmatrix}=0$.

$\therefore \lambda^2+a=0$.

~~$\text{Det}\,A=a$. $\lambda^2=a$.~~

$\therefore \lambda^2=-a$.

$\therefore \lambda=\pm i\sqrt{a}$

(可代入一个a值).

③ $\because (A-\lambda I)v=0$.

$\therefore A-\lambda I$

$=\begin{bmatrix}0 & -1\\ a & 0\end{bmatrix}-\begin{bmatrix}\pm i\sqrt{a} & 0\\ 0 & \pm i\sqrt{a}\end{bmatrix}$.

对于 $\lambda=i\sqrt{a}$.

$\therefore A-\lambda I=\begin{bmatrix}-i\sqrt{a} & -1\\ a & -i\sqrt{a}\end{bmatrix}$.

$\therefore \begin{bmatrix}-i\sqrt{a} & -1\\ a & -i\sqrt{a}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$.

④ $\therefore \begin{bmatrix}v_1\\ v_2\end{bmatrix}=\begin{bmatrix}1\\ -i\sqrt{a}\,v_1\end{bmatrix}$.

对于 $\lambda=-i\sqrt{a}$. ⑤

.....

(其中eigenvalue的实部为增长或衰减, 虚部为频率)

或者Summary

↓

(eigenvector为幅度与相位关系)

⇓

⑪可用MATLAB+反面求的nonlinear center可得数值解来画图.

$\dot{x}=(a-bx)x$.

↳ 根据文章/题目列式 → x与y互相影响.

$$\begin{cases}\dot{x}=(a-bx)x-cxy\\ \dot{y}=(d-hy)y-fxy.\end{cases}$$

② 根据题目的要求 → assumption.

i.e. In $\dot{x}$, $a>0$, $b=0$, $c>0$.

In $\dot{y}$ $d<0$, $h=0$, $f<0$.

③ Simplifying with Hare & Lynx background.

$$\begin{cases}\dot{H}=aH-HL.\\ \dot{L}=-L+HL\end{cases}$$

④ Find Critical / balanced point. (where $\dot{x}=0$).

i.e. $\dot{H}=0$, $\dot{L}=0$.

$\therefore H(a-L)=0$. & $L(H-1)=0$.

$a=L$ 或 $H=0$ $\qquad$ $H=1$ 或 $L=0$.

$\therefore (H,L)=(0,0)$ 或 $(1,a)$ ← stable.

↓ ↓

unstable. 灭绝 平衡.

⑤ 找近似值, 设为 (h,l).

则 $H=1+h$ & $L=a+l$, 且 $\leq$ 值很小 $\therefore \dot{H}=\dot{h}$ & $\dot{L}=\dot{l}$

⑥ 目标: 求 eigenvalue & eigenvector.

$\therefore \begin{bmatrix}\dot{h}\\ \dot{l}\end{bmatrix}=A\begin{bmatrix}h\\ l\end{bmatrix}$.

↓

~~Jacobian~~ Linearization matrix.

⑦ $\because J=\begin{bmatrix}\frac{\partial \dot{H}}{\partial H} & \frac{\partial \dot{H}}{\partial L}\\ \frac{\partial \dot{L}}{\partial H} & \frac{\partial \dot{L}}{\partial L}\end{bmatrix}$ $\qquad$ $\therefore J=\begin{bmatrix}a-L & -H\\ L & -1+H\end{bmatrix}$.

~~⑧ 代入 $H=1+h$, $L=a+l$~~

~~$H=a+b$~~

⑧ 代入 $(1,a)$ 进]

$\therefore A=\begin{bmatrix}0 & -1\\ a & 0\end{bmatrix}$.

Project 2.

3×3 Lotka-Volterra.

i.e. ① x=elephant y=rhino z=Antelope.

$$\dot{x} = x(1-x) - \boxed{a}xy - \boxed{b}xz.$$
$$\dot{y} = y(1-y) - \boxed{b}xy - \boxed{a}yz.$$
$$\dot{z} = z(1-z) - \boxed{a}xz - \boxed{b}yz.$$

where $b>a>0$.
i.e. elephant eats more grass than rhino, but they are competitors.

② Finding Critical point.
Total: $2^3=8$ Critical points.
① $x=y=z=0 \Rightarrow \begin{bmatrix}0\\0\\0\end{bmatrix}$.
② $y=z=0, x=1 \Rightarrow \begin{bmatrix}1\\0\\0\end{bmatrix}$ + 2 more.
③ ~~x~~ $z=0 \Rightarrow \frac{1}{1-ab}\begin{bmatrix}1-a\\1-b\\0\end{bmatrix}$ + 2 more.
④ $x,y,z\neq 0 \Rightarrow$ MATLAB.

(M1 is 手动版) → M2 ③ $f \approx f(a) + J(a)(x-a)$.
$\vec{f} = \begin{pmatrix}f_1\\\vdots\\f_m\end{pmatrix}$ & $\vec{x} = \begin{pmatrix}x_1\\\vdots\\x_m\end{pmatrix}$ & $\vec{a}\begin{pmatrix}a_1\\\vdots\\a_m\end{pmatrix}$.

How to find the J (Jacobian matrix)?

$$\begin{cases}\dot{x} = x(1-x) - axy - bxz \Rightarrow f(x,y,z)\\ \dot{y} = y(1-y) - bxy - ayz \Rightarrow g(x,y,z).\\ \dot{z} = z(1-z) - axz - byz \Rightarrow h(x,y,z).\end{cases}$$

$\therefore J(x,y,z) = \begin{bmatrix}f_x & f_y & f_z\\ g_x & g_y & g_z\\ h_x & h_y & h_z\end{bmatrix}$. (where $f_x = \frac{\partial f}{\partial x}$)

* $\dot{x} = x - x^2 - axy - bxz$
$\dot{y} = y - y^2 - bxy - ayz$
$\dot{z} = z - z^2 - axz - byz$

Summary

$$J(x,y,z) = \begin{bmatrix}1-2x-ay-bz & -ax & -bx.\\ -by & 1-2y-bx-az & -ay.\\ -az & -bz & 1-2z-ax-by\end{bmatrix}$$

④. eigenvalue & vector.

Project 3

$\vec{F}_G$ - Gravity.

$\vec{T}_L$ - Tangent to Cable at right.

$\vec{T}_R$ - Tangent to Cable at right.

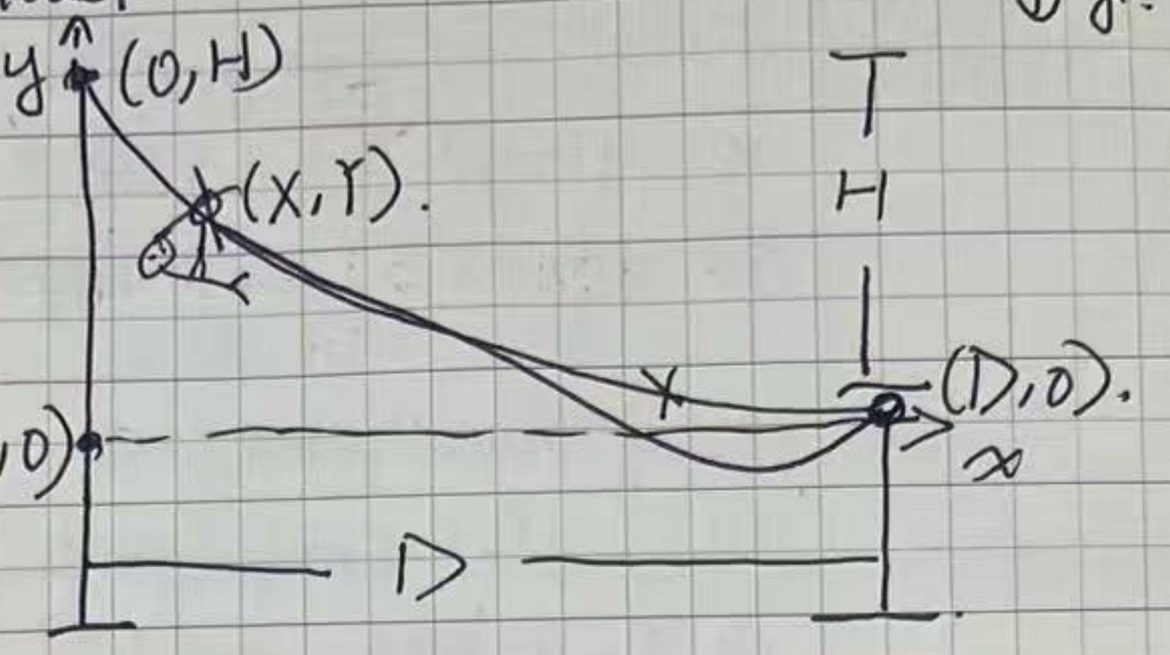
$\vec{F}_D$ - Drag force.

Air resistance force, which $|\vec{F}_D| = c|\vec{v}|$

c: drag coefficient i.e. shape of rider/density of air.

Zipline.

① Model



② Two components for equation

1. Cable. (索道).

L = length = $\sqrt{D^2+H^2}$ (无人时为(0,H)到(D,0)直线距离)

λ = mass/length.

2. Rider.

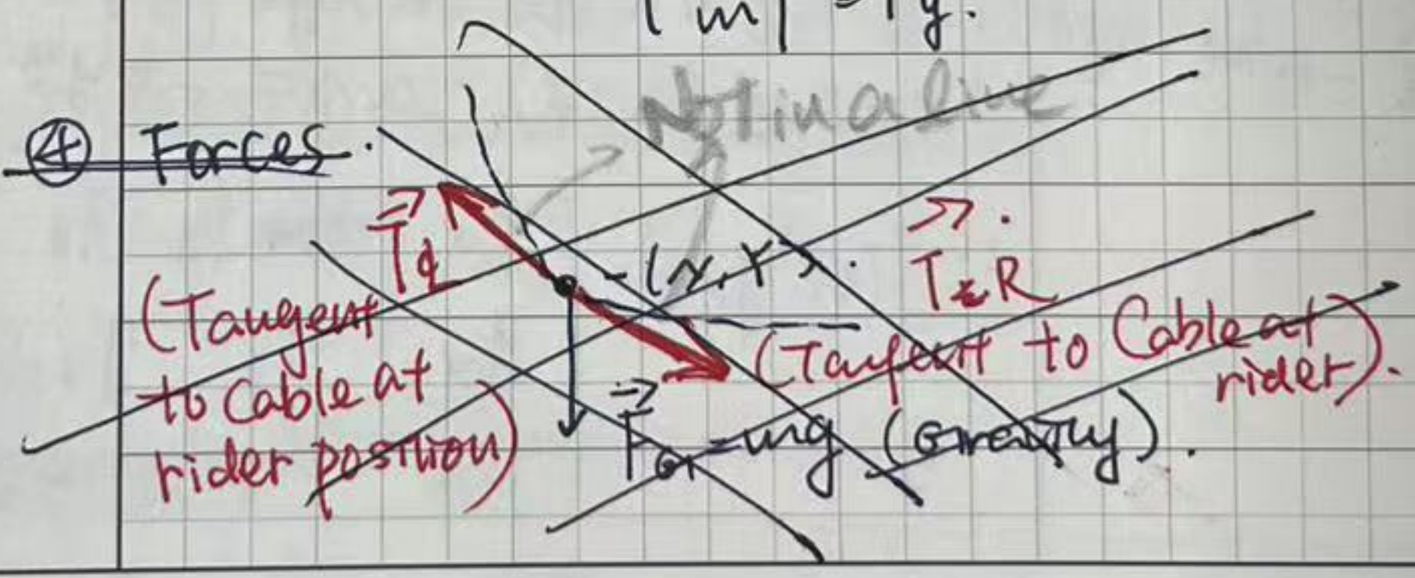
m = mass. (~~count~~ combined mass)

③ Goal:

Find $(X(t), Y(t)) = \vec{X}(t)$ (time)

$\because m \cdot \ddot{\vec{X}} = \vec{F} \rightarrow \begin{cases} m\ddot{X} = F_x \\ m\ddot{Y} = F_y. \end{cases}$

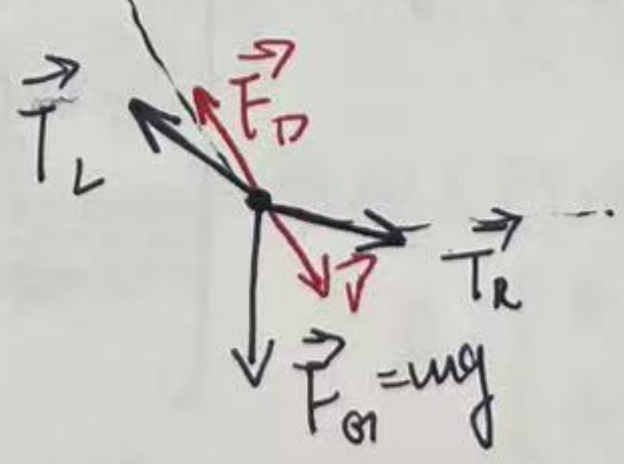
④ ~~Forces~~.



Summary ④ Forces.

* Assume: $|\vec{T}_L| = |\vec{T}_R| = T$.

magnitude same.



$(y'')^2 \frac{1}{A^2}(1+(y')^2)$.

左求导:

$\frac{d}{dx}[(y'')^2]$.

let $y''=u$.

$\therefore \frac{d}{dx}(u^2) = 2u\frac{du}{dx}$.

$\therefore y''=u$

$\therefore \frac{d}{dx}(y'') = \frac{du}{dx}u$.

$\therefore y''' = \frac{du}{dx}$

$\therefore \frac{d}{dx}(y'')^2$

$= 2y''\cdot y'''$

右边求导:

$\frac{d}{dx}\left[\frac{1}{A^2}(1+(y')^2)\right]$

$= \frac{1}{A^2}\cdot\frac{d}{dx}(1+(y')^2+2y')$

$= \frac{1}{A^2}\cdot(2y''y+2y'')$

$= \frac{1}{A^2}\frac{d}{dx}\cdot 2y'\cdot y''$

Summary

$\therefore 2y''\cdot y''' = \frac{1}{A^2}\cdot 2y'\cdot y''$ if $y''\neq 0$.

$y''' = \frac{1}{A^2}\cdot y'$

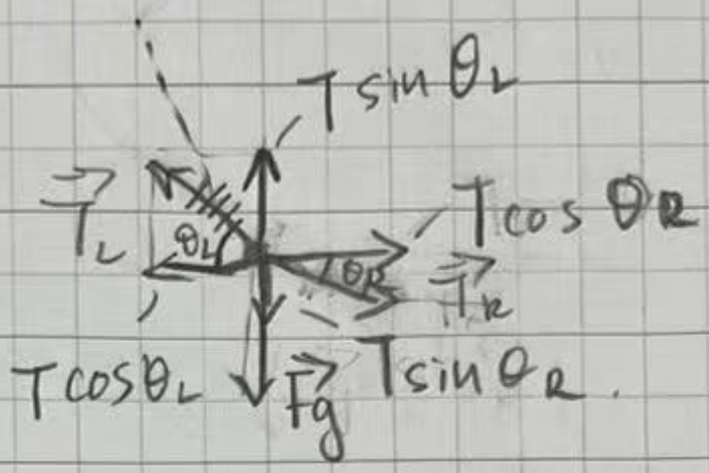
$\therefore y''=0$ or $y''' = \frac{1}{A^2}\cdot y'$

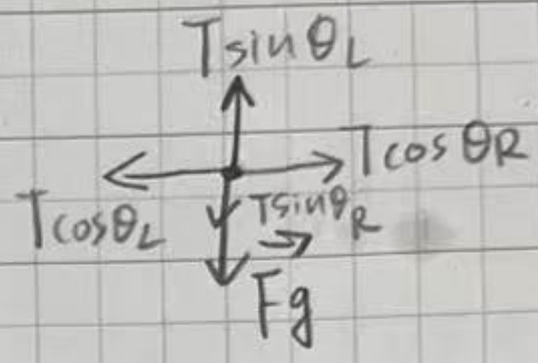
① $\therefore$ if $y''=0$ then $ax+b=y$

② let $y'=u$.

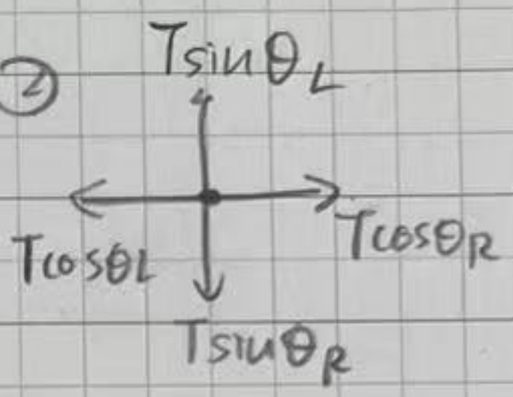
$\therefore u'' = \frac{1}{A^2}u$

⑤ Listing the partial equation - (Analyze force).

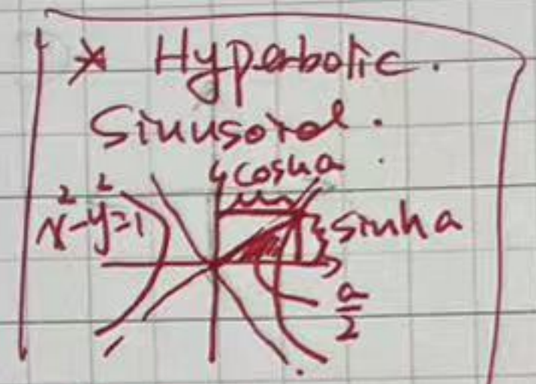




① 

② 

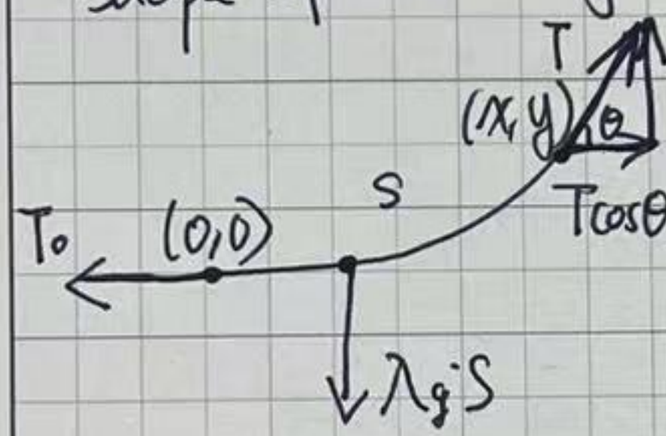
* Hyperbolic.
Sinusoid.



↳ $u = \sinh\left(\frac{x-C_1}{A}\right)$

⑥ Modelling the cables.

- Shape of the hanging cable.



m (of cable) = $\lambda \cdot s$ (length).

$\therefore \frac{\text{Vertical}}{\text{Horizontal}} = \frac{\lambda g}{To}\cdot s = \tan\theta$.

↳ let be $\frac{1}{A}$

$\therefore \frac{s}{A} = \tan\theta$.

$To = T\cdot$ $\lambda g\cdot s = T\sin\theta$.

Vertical: ~~$To = T\cos\theta$~~.

$\Rightarrow$ Horizontal: $To = T\cos\theta$

$\therefore \tan\theta = f'(x) = \frac{dy}{dx}$.

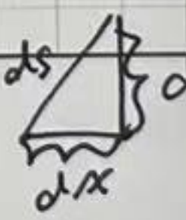
$\therefore = y'$

$\therefore \frac{s}{A} = y'$

$\therefore d(\frac{s}{A}) = d(y')$

$\therefore \frac{1}{A}\cdot ds = y''dx$.

constant

$\therefore$ 

$\therefore ds = \sqrt{dx^2+dy^2}$

$\therefore ds = \sqrt{1+(y')^2}\cdot dx$

$\therefore \frac{1}{A}\cdot\sqrt{1+(y')^2} = y''$

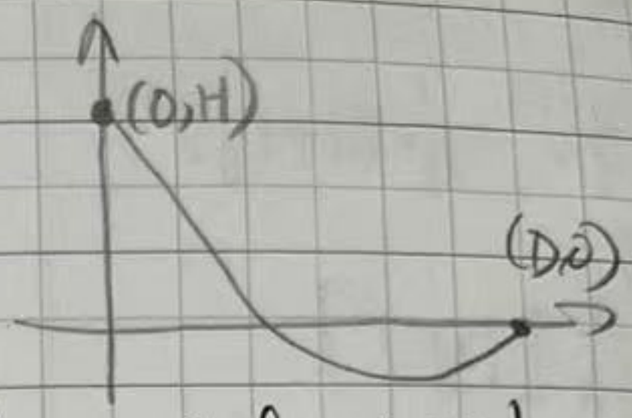
∴ $u = \sinh\left(\frac{x-C_1}{A}\right)$.

∵ A is constant.

∴ $u = \sinh\left(\frac{x-C_1}{C_2}\right)$.

∵ $y' = u$.

∴ $y = C_3 + C_2 \cosh\left(\frac{x-C_1}{C_2}\right)$



∴ { 1. length is L,
2. pass (0,H), (D,0)

→ Boundary conditions.

∴ $C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$ & $y(C,0) = H$ → Means: when $x=0$, y is H.

$y(C,D) = 0$.

length $(y(C)) = L$.

→ 符合zipline形状

No rider ← ↳ 绳子自然悬挂的形状

⑦. Add a ~~static~~ stationary rider

⑧ Motion of rider.

$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ → ~~to~~ fsolve → Solve the shape at that instant.

↓

~~Store~~ Find corresponding $T(t)$, $\theta_R(t)$, $\theta_L(t)$. — 根据曲率.

↓

Use ODE 45 绳子向上受力到加速度 — 分力行合力

↓

$\begin{pmatrix} x(t+\Delta t) \\ y(t+\Delta t) \end{pmatrix}$

Summary